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Center vortex properties in the Laplace-center gauge of $SU(2)$ Yang-Mills theory

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Abstract

Resorting to the the Laplace center gauge (LCG) and to the Maximal-center gauge (MCG), respectively, confining vortices are defined by center projection in either case. Vortex properties are investigated in the continuum limit of $SU(2)$ lattice gauge theory. The vortex (area) density and the density of vortex crossing points are investigated. In the case of MCG, both densities are physical quantities in the continuum limit. By contrast, in the LCG the piercing as well as the crossing points lie dense in the continuum limit. In both cases, an approximate treatment by means of a weakly interacting vortex gas is not appropriate.

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Introduction.

It was already proposed in the late seventies that the essence of quark confinement can be most easily grasped in certain gauges where topological degrees of freedom act as confiners. Color magnetic monopoles [1]-[5] and vortices [6, 7] are considered as candidates for such relevant infrared degrees of freedom.

The vortex picture of confinement received a recent boost when the maximal center gauge was introduced and when the vortices were identified by so-called center projection [8, 9, 10].

Let $U_\mu(x)$ denote the link variables of the $SU(2)$ lattice gauge field theory. The maximum center gauge fixing is defined by maximizing the functional

$$S_{\text{MCG}}[U^\Omega] = \sum_{x,\mu} [\text{tr} U_\mu^\Omega(x)]^2 \rightarrow \max, \\ U_\mu^\Omega(x) = \Omega(x) U_\mu(x) \Omega^\dagger(x + \mu) \quad (1)$$

with respect to all gauge transformations $\Omega(x)$. This fixes the gauge upto center gauge transformations $\Omega(x) = Z(x)$, $Z(x) \in \mathbb{Z}_2, \forall x$. Center projection, which amounts to replace each fundamental link by its nearest center element, provides a well defined (but gauge dependent) prescription for extracting the center vortex content of a field configuration: in the center projected theory plaquettes equal to a non-trivial center element represent center vortices and are referred to as P-vortices. Numerical simulations provide evidence, that P-vortices identified after center projection in the maximal center gauge are relevant infrared degrees of freedom in the following respects:

1. Calculation of the Wilson loop from the P-vortices provides the full string tension [8, 9, 10, 11]
2. The area density of the P-vortices shows the proper scaling behavior [12, 13]
3. When the P-vortices are removed from the lattice configurations chiral symmetry is restored [14].

Unfortunately, in the maximal center gauge fixing one encounters a “practical” Gribov problem: the numerical algorithms fail to find the global maximum of the gauge fixing functional equation (1). If one performs random gauge transformations N_{copy} times before employing the maximum finding algorithm while monitoring the maximum value of S_{fix} , one finds that the string tension calculated from center projected configurations significantly

depends on N_{copy} [15], at least for small lattice sizes. A recent investigation shows that this spurious dependence is lessened if the lattice size is chosen larger than the average vortex size [16]. The practical Gribov problem naturally appears in variational gauges and, in particular, is also inherent in the maximal abelian gauge [17].

To circumvent the (practical) Gribov problem, the Laplace version [18] of the center gauge was introduced in [19]. In this case, the numerical task boils down to the calculation of the two eigenvectors of a lattice matrix. Present days algorithms and computational power allow for an unambiguous gauge fixing for the lattice sizes under considerations.

In this letter, we investigate whether one can attribute continuum physics to the vortices in Laplace center gauge. For this purpose, we study properties such as vortex area density, density of vortex crossings and vortex cluster size distribution and contrast these results with the ones obtained from vortex configurations obtained in maximal center gauge.

Laplace center versus maximal center gauge fixing.

In order to make the letter self-contained we briefly review the Laplace version [18] of the center gauge [19, 20] which is defined as follows: one finds the two lowest eigenvectors $\phi_a^{(1)}(x), \phi_a^{(2)}(x)$ of the adjoint covariant Laplacian

$$\Delta_{x,y}^{ab} [\hat{U}] = 2a\delta_{x,y}\delta^{ab} - \sum_{\pm\hat{\mu}} \hat{U}_{\mu}^{ab}(x)\delta_{x\pm\hat{\mu},y}, \quad (2)$$

where a denotes the lattice spacing. Since $\hat{U}_{\mu}(x)$ is center blind the eigenvectors of $\Delta [\hat{U}]$ remain unchanged under center gauge transformations. Note also, that the adjoint link $\hat{U}_{\mu}(x)$ is a real symmetric 3×3 (adjoint) color matrix, so that at fixed x the eigenvectors $\phi(x)$ of eq. (2) have 3 real components $\phi_{a=1,2,3}(x)$ and thus form an ordinary 3-dimensional vector $\vec{\phi}(x)$ in (adjoint) color space.

In a first step the gauge freedom is exploited to rotate the lowest eigenvector $\vec{\phi}^{(1)}(x)$ parallel to the 3-axis. This fixes the gauge up to abelian gauge rotations around the 3-axis in color space and defines the so-called Laplace abelian gauge. Obviously this abelian gauge is ill-defined at those points x in space, where $\phi^{(1)}(x) = 0$, which defines the positions of magnetic monopoles. In a second step the residual $U(1)/\mathbb{Z}_2$ gauge freedom is fixed by gauge rotating the next to lowest eigenvector $\vec{\phi}^{(2)}$ into the 1 – 3 – plane. This is equivalent to rotating the component $\vec{\phi}_{\perp}^{(2)}$ of $\vec{\phi}^{(2)}$, which is orthogonal to the

3-axis, parallel to the 1-axis. Obviously this gauge fixing procedure is ill-defined, when $\vec{\phi}^{(1)}(x)$ and $\vec{\phi}^{(2)}(x)$ are parallel, so that (after Laplace abelian gauge fixing) $\vec{\phi}_\perp^{(2)}(x) = 0$. The latter condition involves two constraints, so that these gauge singularities have co-dimension 2. They form (closed) lines in $D = 3$ and sheets in $D = 4$ and hence represent vortex singularities. Indeed, one can show by going around the vortex singularities (where $\vec{\phi}_\perp^{(2)}(x) = 0$) that the holonomy in the fundamental representation acquires a non-trivial center element, see also [21]. This justifies to call these gauge singularities center vortices.

Unfortunately the solutions of the equations $\vec{\phi}^{(1)}(x) = 0$ and $\vec{\phi}_\perp^{(2)}(x) = 0$ (which define the positions of magnetic monopoles and vortices) are ambiguously defined in the lattice formulation due to the finite discretization of space-time. In the lattice case, a practical approach to identify the center vortices is to use center projection on top of Laplace center gauge fixing. In this way one avoids the Gribov problem of the maximal center gauge fixing, but keeps the simplicity of the identification of the vortex content of the center projection. One should stress however, that a priori the positions and properties of the resulting center vortices need not to coincide with those defined by the gauge singularities occurring for parallel color vectors $\vec{\phi}^{(1)}(x)$ and $\vec{\phi}^{(2)}(x)$. It is the aim of the present paper to study the properties of center vortices arising after center projection on top of Laplace center gauge fixing and to compare them with those arising after center projection in the maximal center gauge.

Vortex properties – comparison of gauges.

Here, we will compare the properties of the vortices arising from the Laplace center gauge (LCG) with those obtained in maximal center gauge (MCG). In the latter case, we check for the practical Gribov problem by using two different methods to implement the MCG: firstly, we use a naive over-relaxation method (100 sweeps) with no further effort to find the global maximum of the gauge fixing condition. The extracted data in the figures below are labeled **max**. Secondly, we firstly implement the LCG and adopt the over-relaxation method afterwards (label **lap-max**). Different numerical data in either case correspond to different Gribov copies in configuration space. We stress that we here do not attempt to contribute to the ongoing discussion on the practical Gribov problem (see [15] and [16]). Rather than, we are interested in the gross features of the vortex properties arising in MCG and LCG, respectively. It will turn out below that the scaling behavior (and therefore their relevance

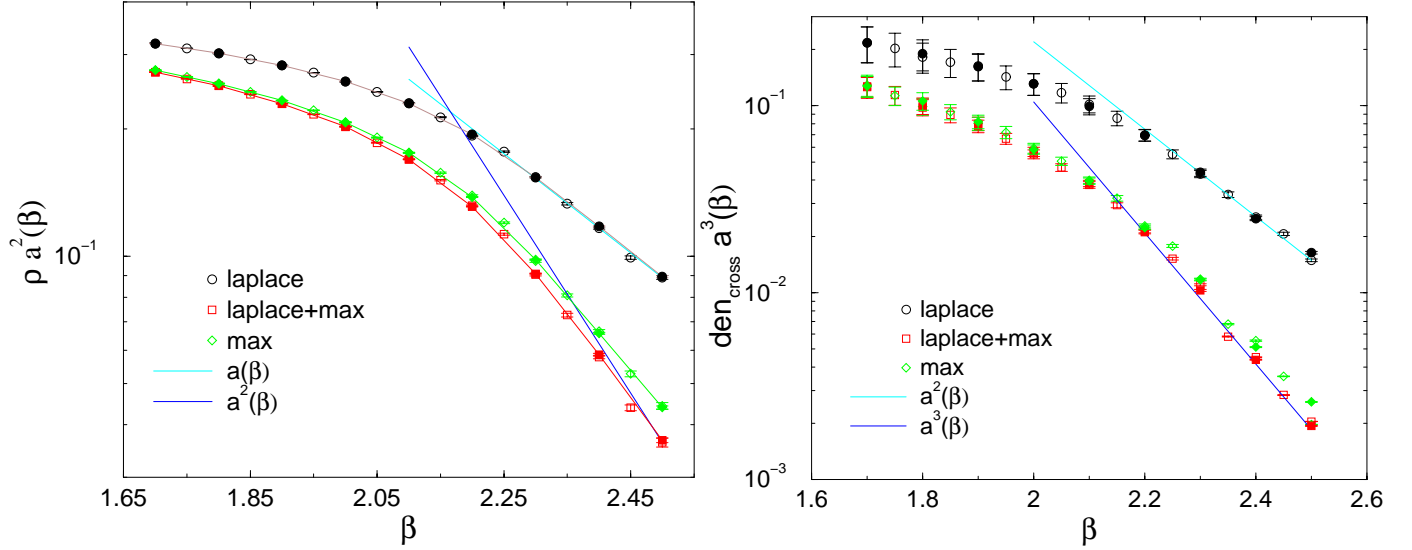


Figure 1: Scaling of the planar density of vortex intersection points with a given space-time plane (left panel), and scaling of the vortex crossing points (right panel). Open symbols: 12^4 lattice; full symbols: 16^4 lattice;

for the continuum limit) is different, and that this difference largely exceeds the effect by Gribov noise.

In a first step, we analyzed the vortex area density (figure 1). While in the case of the MCG, the numerical data are consistent with a physical vortex area density, the numerical data in the case of the LCG indicate that the points where vortices pierce a 2-dimensional hyperplane of the lattice lie dense in the continuum limit.

To be precise, let $F \in \mathbb{R}^2$ be the set of points where the LCG vortices pierce a given 2-dimensional hypersurface. A ϵ -cover of F is a union of *non-empty* subsets V_i with maximal extension $|V_i| = \sup\{|x - y| : x, y \in V_i\}$ less than ϵ which cover F , i.e. $F \subset \bigcup_i V_i$. The minimal number of subsets which is required for the ϵ -cover of all piercing points, i.e. F , is denoted by N_ϵ . For sufficiently small ϵ one finds that

$$N_\epsilon \propto \epsilon^{-d} . \quad (3)$$

with d being the Hausdorff dimension. In practice, we choose $\epsilon = a(\beta)$ (a lattice spacing) and investigate the dependence of N_ϵ on ϵ , i.e. a , by varying β . One finds

$$N_\epsilon = \rho \mathcal{V}_2 = \kappa \frac{\mathcal{V}_2}{a^2} ; \quad \kappa := \rho a^2 , \quad (4)$$

where \mathcal{V}_2 (fixed) is the physical volume of the 2-dimensional hypersurface. Hence, the scaling of the vortex area density in units of the lattice spacing, i.e. κ , is related to the Hausdorff dimension. The numerical data for κ (see figure 1) are well fitted within the scaling window $\beta \in [2.1, 2.5]$ by

$$\begin{aligned}\kappa &= \exp\{-A\beta + B\} \propto \epsilon^{2-d} \\ A &= 2.25 \pm 0.02, \quad B = 3.30 \pm 0.05,\end{aligned}$$

(only statistical errors are presented). Assuming that 1-loop scaling is appropriate for the above range of β , i.e.

$$\epsilon \equiv a(\beta) \propto \exp\left\{-\frac{3\pi^2}{11}\beta\right\},$$

the Hausdorff dimension can be phrased in terms of the coefficient A , i.e.

$$d = 2 - \frac{11}{3\pi^2}A, \quad d = 1.16 \pm 0.01. \quad (5)$$

The points where the P-vortices of the LCG pierce the 2-dimensional hypersurface almost form 1-dimensional structures. This should be compared with the P-vortices of the MCG which have a vanishing Hausdorff dimension since ρ is independent of $a(\beta)$.

It was already observed in [20] that the density of the points where LCG vortices pierce the hyperplane lack an interpretation as isolated points in the continuum limit. Rather than disqualifying this density as pure lattice artefact [20], our finding that the density scales according $1/a(\beta)$ suggests the physical interpretation that, in the continuum limit, the LCG vortex material exists in a condensed phase.

An analogous analysis can be performed for the density of crossing points of P-vortices where vortex lines intersect each other in a given spatial hypercube. Now, the set of points F is part of the \mathbb{R}^3 , and the subsets V_i may be viewed as balls of radius ϵ . In the case of the MCG, the numerical data for the density of intersection points is well fitted by (16^4 lattice, $\beta \in [2.1, 2.5]$)

$$\begin{aligned}\rho_{\text{den}}^{\text{max}} a^3 &= \exp\{-A_{\text{max}}\beta + B_{\text{max}}\}, \\ A_{\text{max}} &= 8.079 \pm 0.026, \quad B_{\text{max}} = 13.96 \pm 0.06.\end{aligned}$$

These results yield the Hausdorff dimension

$$d_{\text{den}}^{\text{max}} = 3 - \frac{11}{3\pi^2}A_{\text{max}}, \quad d_{\text{den}}^{\text{max}} = 0.00 \pm 0.01. \quad (6)$$

When this analysis is repeated for the case of the LCG, we find

$$\begin{aligned} A_{\text{lap}} &= 7.079 \pm 0.05 , & B_{\text{lap}} &= 11.64 \pm 0.11 , \\ d_{\text{den}}^{\text{lap}} &= 0.37 \pm 0.02 . \end{aligned} \quad (7)$$

While the crossing points in the MCG indeed form isolated points in three dimensions with a physical density (cf. detailed discussion below), the corresponding points of the LCG lie dense with a Hausdorff dimension being significantly different from zero.

Random thin vortex model.

In order to interpret information carried by the density of P-vortex crossing points (as defined in the previous section), it is instructive to compare the numerical results obtained for P-vortices in MCG and in LCG with the predictions of a model of randomly distributed thin vortices. It should be stressed that the thin random vortex ensembles considered in the following are conceptually distinct from effective infrared models describing random surfaces of thick vortices such as have recently been employed [22] to describe infrared properties of the Yang-Mills ensemble. In the latter vortex model, spatial correlations, such as the crossing point density, are only defined on scales coarser than the vortex thickness.

Let us consider a 3-dimensional slice through the 4-dimensional lattice universe. In this 3-dimensional hypercube vortices appear as closed loops on the dual lattice. Let a be the lattice spacing, N_S be the total number of sites and N_L be the number of links occupied by vortices. The vortex area density in units of the lattice spacing, i.e. ρa^2 , is related to the probability that a single vortex goes through a plaquette and is given by

$$\rho a^2 = p = \frac{N_L}{3N_S} . \quad (8)$$

Due to the \mathbb{Z}_2 Bianchi identity (which ensures, that the vortex loops are closed) the number of links occupied by vortices and attached to a single site is even. For a dilute vortex gas, we neglect the constraint imposed by the Bianchi identity, and approximate the probability that two vortices intersect at a given site of the dual lattice by

$$p_{\text{cross}} \approx p^2 = \rho^2 a^4 . \quad (9)$$

Let N_{cross} be the number of sites, where two vortices cross, i.e. $N_{\text{cross}} = p_{\text{cross}} N_S$. Then the density of sites, where two vortices cross, is given by

$$\rho_{\text{cross}} = \frac{N_{\text{cross}}}{V} = \frac{p_{\text{cross}}}{a^3} \approx \rho^2 a . \quad (10)$$

It has been shown that for the P-vortices obtained after center projection in maximal center gauge the vortex (area) density ρ is a physical quantity and thus independent of the lattice spacing. This is also seen in fig. 1, which shows that the density ρa^2 of P-vortices arising from MCG fixing scales like a^2 . For fixed vortex (area) density ρ , the random thin vortex model yields a vanishing density of crossing points when we approach the continuum limit $a \rightarrow 0$. This is expected since in the continuum two lines generically do not intersect.

By contrast, our lattice simulations show that the density ρ_{cross} for the P-vortices in MCG is a physical quantity independent of a (see fig. 1 right panel) and non-vanishing. Thus, in the continuum limit the P-vortex ensemble in MCG possesses information which cannot be described in terms of the random thin vortex model.

In the case of the P-vortices of the LCG, neither the vortex area density ρ nor the density of crossing points ρ_{cross} is independent of the lattice spacing a . In fact, both quantities diverge like $1/a$ in the continuum limit $a \rightarrow 0$ (see figure 1). Note, however, that this behavior is consistent with a random thin vortex model: if we consider a random thin vortex model with area density $\rho \sim 1/a$, eq. (10) indeed implies that $\rho_{\text{cross}} \sim 1/a$.

To summarize we find, that both the P-vortex (area) density ρ and the density of P-vortex crossing points are physical quantities (in the sense that they show proper renormalization group scaling) in the MCG and unphysical ones in the LCG, respectively. In the case of MCG, the behavior of ρ and ρ_{cross} cannot be understood on the basis of a random thin vortex model with a fixed density as sole input. In the case of LCG, the ratio of ρ and ρ_{cross} is compatible with the prediction of a random thin vortex model.

Vortex cluster properties.

In order to reveal the mechanism which leads to the different behavior of vortex area density ρ and the density of crossing points ρ_{cross} , in MCG and LCG, respectively, it is necessary to investigate in either case the properties of vortex clusters embedded in a 3-dimensional hypercube. One such quantity of interest is the probability distribution of the size of the vortex cluster. The numerical result for a 16^4 -lattice and $\beta = 2.4$ is shown in figure 2. Small size clusters are most numerous in both cases, MCG and LCG. However, these small size clusters do not contain much vortex material as one finds by considering probability $p(s)$ that an elementary vortex link belongs to

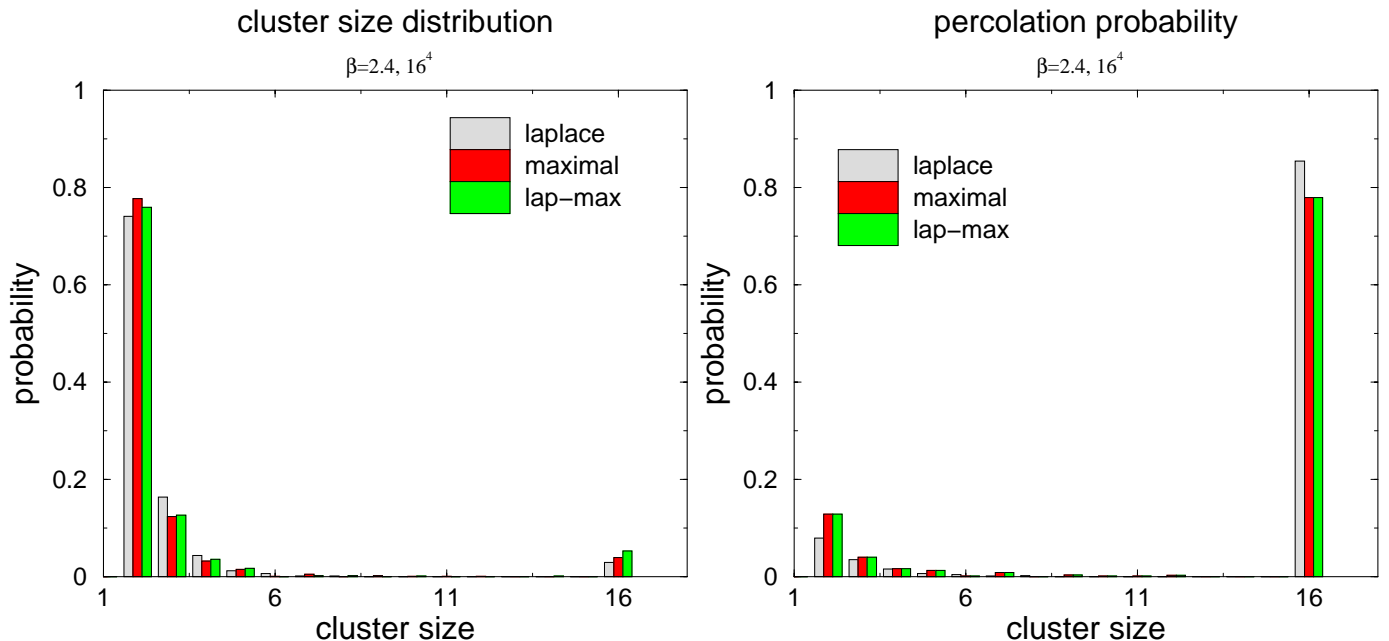


Figure 2: Cluster size distribution (left panel), and percolation probability (right panel).

a cluster of size s (percolation probability). At small temperatures, the situation that almost all the vortex material is stored in a cluster of the size of the lattice universe occurs most frequently (see figure 2) (see also [11]). We therefore find that the probability distributions of both cases, MCG and LCG, do not show large differences. In the case of the LCG, there is a slight enhancement of the percolation probability for maximum size clusters. This indicates that the excess of vortex material in the case of LCG is part of the huge size clusters, rather than attributed to additional small isolated vortex cluster.

Conclusions.

The Laplace-center gauge (LCG) is superior to other variational gauges since ambiguities stemming from the numerical implementation of the gauge are avoided. We have investigated properties of vortices arising by center projection after implementing the maximal center gauge (MCG) and the LCG, respectively. We have studied the density of points where vortices pierce a 2-dimensional hyperplane, and we have presented results for the density of points where the vortices intersect within a spatial hypercube.

In the case of the MCG, both densities properly scale towards the continuum limit. Since a random thin vortex model with given area density as input predicts a vanishing density of intersection points, the vortex ensemble possesses correlations not present in a weakly interacting vortex gas (cf. also [13]).

In the case of the LCG, the above densities diverge in the continuum limit. Given that the string tension is well reproduced [19, 14] from the LCG vortex ensemble, the excess of vortex material is due to small size vortex fluctuations. Further informations are provided by the vortex cluster properties, size distribution and percolation probability. The picture of large percolating vortex clusters with enhanced UV vortex fluctuations, which cannot influence the string tension, is consistent with our numerical findings.

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